ON THE DESIGN CURVES FOR BUCKLING PROBLEMS

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INTRODUCTION

The paper deals with the design curves for the member buckling problems included in many modern standards as the basis for the stability design of steel structures. In the Eurocode 3 [1] member buckling curves are used for the two basic cases: the flexural buckling of columns and the lateral-torsional buckling of beams (LTB). These two buckling curves are of quite high concern and necessary for the standard design of general member stability, therefore the accurate theoretical and experimental verification of them is very important. The column buckling has been researched comprehensively for many years as the basic and simplest case of stability problems. The research included a huge number of experimental tests, complete theoretical investigations using both analytical and numerical models, and deep probabilistic examinations [2]. As a main result the multiple column curves concept has been developed, and as the basic design model the Ayrton-Perry formula (APF) was adopted. This formula has the following main advantages: clear mechanical background, simplicity and flexibility; however it is important to note that it is connected directly with the flexural buckling phenomena. The LTB is far more complicated to handle, the experimental research can usually run into modelling difficulties, the analytical description is limited for the most basic cases; nowadays the most appropriate analysis is the numerical simulation [3]. Exploiting the flexibility of the APF this form was chosen as the design curve for LTB adjusting the design parameters carefully so as to be in accordance with the results of the numerical simulations [3]. Although such a way the LTB curves can be considered adequately verified, the theoretical background of the direct connection between the APF and LTB is not clarified so far. The most important drawback of this is connected with the imperfection factor as the main parameter of the APF: the correct form of the generalized imperfection factor is unknown and the appropriate equivalent geometric imperfections can not be defined. The paper shows a possible way to connect directly the LTB and APF by generalizing the APF using a special rule for the definition of the imperfection factor.

1 COLUMN BUCKLING

The APF was applied originally for geometrically imperfect columns loaded by uniform compression, where the load carrying capacity is corresponds to the onset of yielding at the most compressed fibre [4]. It is important to note that this formula neglects the influence of other imperfections (residual stress, eccentricity) and partial plasticity; however these additional effects can be efficiently modelled through the generalized imperfection factor. The maximum total second order lateral displacement of such a simply supported prismatic member assuming sinusoidal imperfections writes:

$$v = \frac{v_0}{1 - N/N_{cr.z}} \qquad (1)$$

where v_0 is the midspan amplitude of the half-sine wave, $N_{cr.z}$ is the elastic critical buckling load about the minor axis and N is the actual compressive force. At the midspan cross-section the most compressed fibre should reach the yield stress (first yield criterion):

$$\frac{N}{A} + \frac{N \cdot v}{W_z} = f_y \qquad (2)$$

where A is the cross-sectional area, W_z is the elastic sectional modulus and f_y is the yield stress. Substituting *Eq. (1)* into *Eq. (2)* the original form of the APF can be written:

$$(\sigma_{cr} - \sigma_b)(f_y - \sigma_b) = \sigma_b \sigma_{cr} \eta \quad (3)$$

where σ_{cr} is the elastic critical compressive stress, σ_b is the actual compressive stress, and $\eta = v_0 \frac{A}{W_z}$ is the generalized imperfection factor. *Eq. (3)* can also be written by the standard notations:

$$\chi^2 + \chi\left(-1 - \frac{1}{\lambda^2} - \frac{1}{\lambda^2}\eta\right) + \frac{1}{\lambda^2} = 0 \quad (4)$$

where λ is the well-known slenderness ($\lambda = \sqrt{\frac{Af_y}{N_{cr.z}}}$), χ is the buckling reduction factor ($\chi = \frac{N}{Af_y}$). The solution of *Eq. (4)* yields the standard form of the column buckling curve.

2 GENERALIZATION OF THE AYRTON-PERRY FORMULA

The derivation of the column buckling curve has distinct steps, which can be generalized in order to be applicable for other buckling problems, in our case for LTB. The final aim is always the determination of the generalized form of *Eq. (4)*, which yields the reduction factor (χ_{gen}) in terms of the slenderness and imperfection factor. The general form of the slenderness writes:

$$\lambda_{gen} = \sqrt{\frac{\alpha_y}{\alpha_{cr}}} \quad (5)$$

where the α_y and α_{cr} are the first order load amplifiers corresponding to the first yield of the cross section and the elastic critical load respectively. This definition is somewhat similar to the one applied in the general method for stability check in EC3 [1], but is not exactly the same, because the final solution corresponds still to the first yield criterion. The steps of the derivation of APF for general stability problems are the following:

1. Determination of the one parameter load and displacement – and corresponding geometrical imperfection – variables, and defining the governing differential equation(s);
2. Applying suitable initial geometric imperfections;
3. Determination of the second order elastic displacements in terms of the load parameter, elastic critical loads (amplification factors) and initial imperfections;
4. Definition of the first yield criterion based on second order section forces which depend on the second order elastic displacements;
5. From the first yield criterion writing the – normally – quadratic equation for the reduction factor in terms of the generalized slenderness and imperfection, and the solution is the appropriate buckling curve.

It is very important to understand, that the crucial point of the general process is step 2, since the proper definition of the shape of imperfections has a great impact on the main equation derived from the first yield criterion (step 4). The main principle of this paper is that if the shape of the initial geometric imperfection is taken exactly the same as the buckling shape (or first eigenshape) of the perfect, elastic member then the equation of step 5 is always has the same form as *Eq. (4)*, the only deviations between the different buckling problems are the concrete definition of slenderness (*Eq. (4)*) and imperfection factor. In the paper this theorem will be proved for LTB of beams showing that the usual simplifications in the shape of initial imperfection do not lead to the form of *Eq. (4)*, therefore the APF can not be constructed.

3 LATERAL-TORSIONAL BUCKLING

Consider the basic case for LTB, which is a simply supported prismatic beam loaded by uniform bending moment about its major axis. In this case the one parameter load is the bending moment *M*,

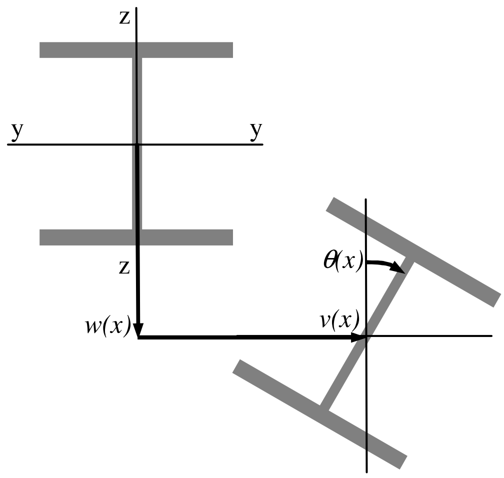

Fig. 1. Displacements of the cross-sections in LTB

the displacements are the two deflections *w(x), v(x)* and section rotation *θ(x)* depending from the axis *x* through the centreline of the member (*Fig. 1*), and the well-known governing system of partial differential equations is the following:

$$\begin{aligned} &EI_y \frac{\partial^2 w(x)}{\partial x^2} = -M \\ &EI_z \frac{\partial^2 v(x)}{\partial x^2} + M\big(\theta(x) + \theta_0(x)\big) = 0 \\ &EI_\omega \frac{\partial^3 \theta(x)}{\partial x^2} - GI_{SV} \frac{\partial \theta(x)}{\partial x} + M\left(\frac{\partial v(x)}{\partial x} + \frac{\partial v_0(x)}{\partial x}\right) = 0 \end{aligned} \tag{6-8}$$

where $v_0(x)$ and $\theta_0(x)$ are the initial imperfections. Since the first equation does not influence the buckling it will be omitted henceforward. Step 2 is now the selection of the suitable shape for the initial imperfections. Omitting the initial imperfections from *Eqs. (7-8)* the solution gives the buckling shape of the perfect member for which the following relationship holds:

$$v_b(x) = \frac{M_{cr}}{N_{crz}} \theta_b(x) \tag{9}$$

where the *b* subscript refers to the buckling shape, and M_{cr} is the elastic critical bending moment. This is the key relationship which should be applied to the shape of initial imperfections further on. In step 3 solving *Eqs. (7-8)* with the general imperfection terms yields the total second order elastic displacements, which takes the following form at the midspan cross-section:

$$\begin{bmatrix} v \\ \theta \end{bmatrix} = \frac{1}{1-\left(M/M_{cr}\right)^2} \begin{bmatrix} 1 & \dfrac{M}{N_{cr.z}} \\ \dfrac{N_{cr.z} M}{M_{cr}^2} & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ \theta_0 \end{bmatrix} \tag{10}$$

This is a general relationship containing a quadratic amplification factor and showing that both initial imperfections influence both displacements. At this point it is a usual simplification to neglect one of the two imperfection terms [5, 6], however following this way the quadratic amplification factor still remains, and the APF in the form of *Eq. (4)* can not be achieved, since this main equation will be cubic instead of quadratic with an untreatably complex solution. So applying the principle introduced in the second section, and accordingly substituting the relationship *Eq. (9)* into *Eq. (10)* one obtains the following form:

$$\begin{bmatrix} v \\ \theta \end{bmatrix} = \frac{1}{1 - M / M_{cr}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ \theta_0 \end{bmatrix} \tag{11}$$

That is the crucial step described in the second section which is resulted in a far simpler relationship and has the same form as *Eq. (4)* obtained for flexural buckling with the same linear amplification factor. The other main conclusion is that if the direction of the vector of initial imperfections is the same as the one of the vector of the buckling displacements then the certain displacements depend on the corresponding imperfection only. Moving on to step 4 the first yield criterion in the midspan cross-section is the following [7]:

$$\frac{M}{W_y} + \frac{M \cdot (v + w\theta) - GI_{SV} \cdot \theta}{W_\omega} + \frac{M \cdot \theta}{W_z} = f_y \tag{12}$$

where W_y, W_z and W_w are the elastic major axis, minor axis and warping sectional moduli respectively, and w is the deflection about the major axis (*Fig. 1*) which can be expressed at midspan as $w = \frac{M \cdot L^2}{2EI_y}$. On the left side of *Eq. (12)* the second and third terms are the normal stresses due to the second order bimoment and minor axis bending respectively. In the second order bimoment the advantageous effect of the Saint-Venant torsional rigidity, and the additional mixed term due to the simultaneous deflection (w) and twist (θ) are considered. Substituting *Eq. (11)* into *Eq. (12)* and applying $\lambda_{LT} = \sqrt{\frac{W_y f_y}{M_{cr}}}$ and $\chi_{LT} = \frac{M}{W_y f_y}$ the quadratic APF for LTB writes:

$$\chi_{LT}^2 \beta_{LT.2} + \chi_{LT} \left(-1 - \frac{1}{\lambda_{LT}^2} - \frac{1}{\lambda_{LT}^2} \eta_{LT} \right) + \frac{1}{\lambda_{LT}^2} \beta_{LT.1} = 0 \tag{13}$$

where the three additional terms are the new generalized imperfection factor, the Saint-Venant torsional rigidity factor and the deflection factor (applying the relationship *Eq. (9)*):

$$\begin{aligned} \eta_{LT} &= v_0 \frac{W_y}{W_\omega} + \theta_0 \frac{W_y}{W_z} = v_0 \left(\frac{W_y}{W_\omega} + \frac{N_{cr.z}}{M_{cr}} \frac{W_y}{W_z} \right) \\ \beta_{LT.1} &= 1 + \theta_0 \frac{GI_{SV}}{W_w f_y} = 1 + v_0 \frac{N_{cr.z}}{M_{cr}} \frac{GI_{SV}}{W_w f_y} \\ \beta_{LT.2} &= 1 - \theta_0 \frac{W_y}{W_\omega} \frac{M_{cr}}{N_{cr.y}} \frac{\pi^2}{2} = 1 - v_0 \frac{W_y}{W_\omega} \frac{N_{cr.z}}{N_{cr.y}} \frac{\pi^2}{2} \end{aligned} \tag{14-16}$$

Using these expressions the buckling curve for LTB can be written as the solution of *Eq. (13)* in the well-known form of the EC3 [1]:

$$\begin{aligned} \chi_{LT} &= \frac{\beta_{LT.1}}{\phi_{LT} + \sqrt{\phi_{LT}^2 - \beta_{LT.1} \beta_{LT.2} \lambda_{LT}^2}} \\ \phi_{LT} &= 0.5 \left[1 + \eta_{LT} + \lambda_{LT}^2 \right] \end{aligned} \tag{17-18}$$

This is the fundamental form of the APF based LTB curve belonging to the first yield criterion of *Eq. (12)* and the specially shaped initial geometric imperfection defined by *Eq. (9)*.

4 DISCUSSION

In order to reveal the most important features and characteristics of the obtained LTB curve – keeping in mind that it is only the basic form belonging to the first yield criterion – let examine deeper the main peculiarities of *Eqs. (17-18)* which are the effects of the special imperfection factors of *Eqs. (14-16)*. In the recent form of EC3 [1] multiple buckling curves are used for LTB, and the distinction is made upon the section type (I-shaped or not), fabrication process (hot-rolled or welded) and in the case of I profiles the height-width ratio (h/b). In this paper we deal with I-shaped profiles and since the effect of fabrication – through the residual stresses – has been omitted only the influence of the geometrical parameters is investigated further. It is found, that it is not solely the h/b ratio what affects significantly the shape of the LTB curves, therefore the examinations are carried out for six different hot-rolled sections (HEAA300, HEA300, HEB300, HEAA900, HEA900, HEB900) in order to show the other types of influential parameters. In *Figs. 2-6* the solid lines represent the 300 sections, the dashed lines represent the 900 sections, and the thickness of the lines show the heaviness of the proper section (thin – HEAA, thick – HEB). All the results are calculated considering a $v_0=L/1000$ initial out-of-straightness and a S235 material. In *Fig. 2* the buckling curves – and the non-dimensional Euler curve –, in *Figs. 3-6* the imperfection factors of *Eqs. (14-16)* are plotted for the six cross-sections against the non-dimensional slenderness.

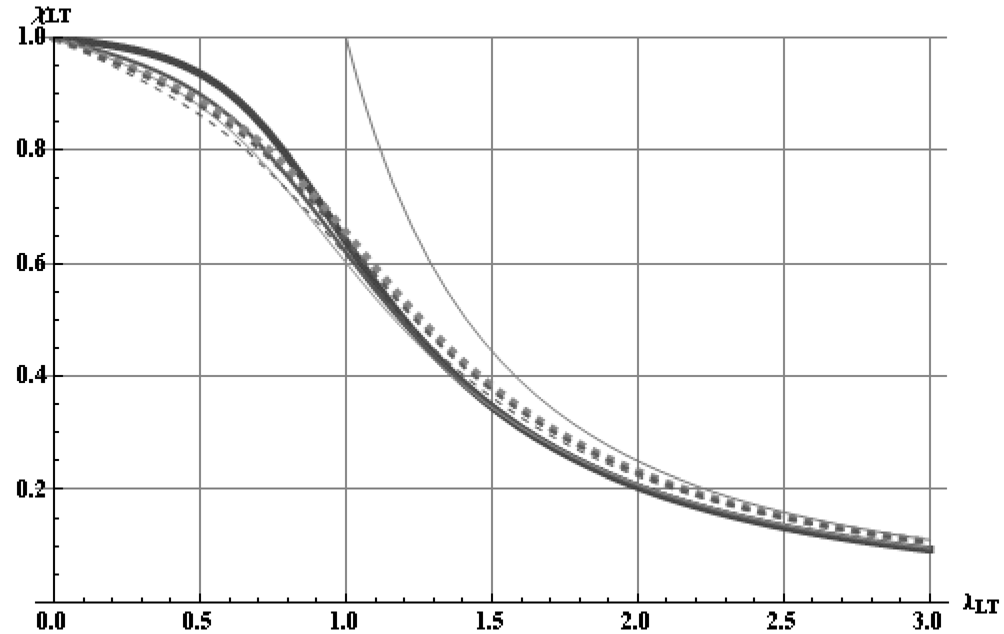

Fig. 2. The LTB buckling curves

From *Fig. 2* it can be first noticed that there is only a little difference between the curves which correspond to higher h/b ratio (~3, dashed lines), and generally these curves take higher values at higher slenderness ($\lambda_{LT} < 1$) and lower values at lower slenderness then the curves correspond to lower h/b ratio (~1, solid lines). It is important here to note that the EC3 distinguish the curves belonging to different h/b ratios only by the generalized imperfection factor (here η_{LT}) and the β factor remains the same for all the cases, this approach does not yield the above difference between the curves. On the other hand there is significant difference between the solid curves at medium slenderness; the curve of the HEB300 section takes higher value with up to 10% then curve of HEAA300 section. This is mainly because of the fact that more compact and heavier profiles have considerably smaller η_{LT} values while the advantageous effect of their Saint-Venant torsional rigidity (β_{LT}) is more significant as it is seen in *Figs. 3-4*. This observation suggests an additional distinction between the heavy and light sections by for example the b/t_f ratio. It is also interesting to remark the effect of the two β_{LT} values in *Figs. 5-6*. Generally the deflection factor $\beta_{LT.2}$ has a dominant influence in case of wider flange profiles (solid lines); while for the higher sections this effect is negligible. It is the result of the much greater deflection along with almost the same minor axis inertia, and that is the reason for the difference between the solid and dashed curves at higher slenderness. This phenomenon suggests that the β factor in the design LTB curves of the EC3 should also depend on the h/b ratio.

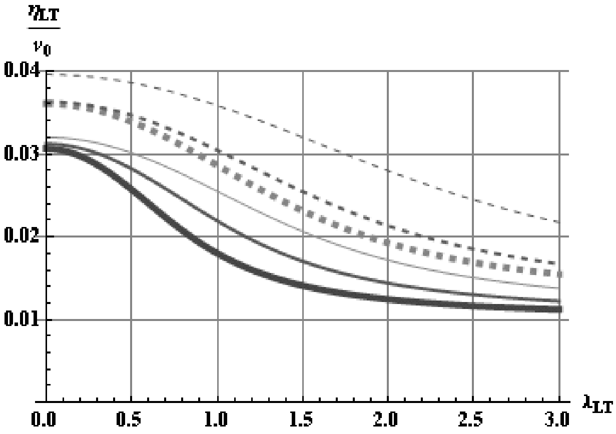

Fig. 3. The generalized imperfection factors

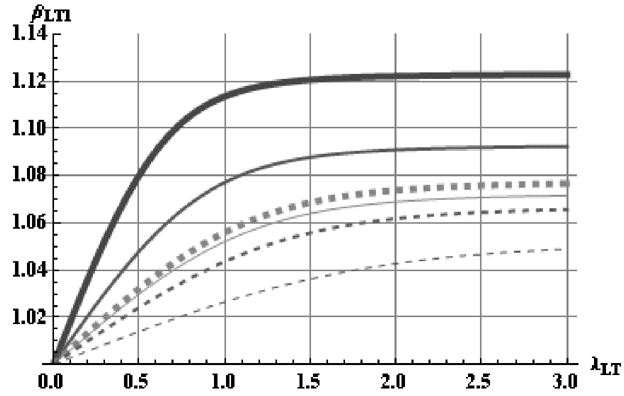

Fig. 4. The Saint-Venant torsional rigidity factors

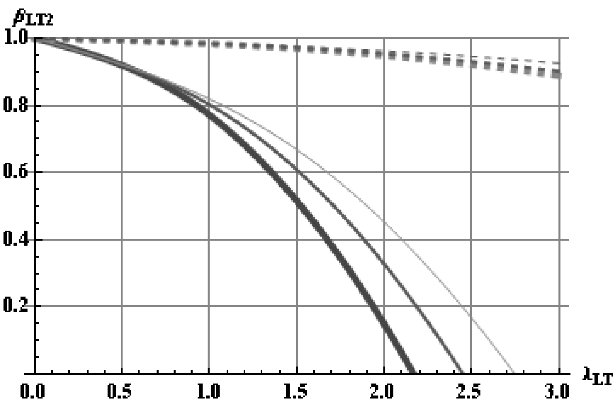

Fig. 5. The deflection factors

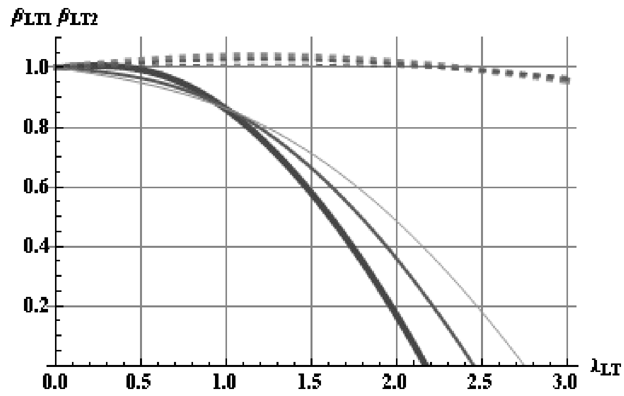

Fig. 6. The combined effect of $\beta_{LT.1}$ $\beta_{LT.2}$

5 CONCLUSION

The main point of the paper was to clarify the mechanical basis of the design curve used for lateral-torsional buckling in the EC3. This aim has been achieved by generalizing the original form of the Ayrton-Perry formula – the basis for flexural buckling design curve – introducing a specific definition of the initial geometric imperfection. By this approach the direct connection between the generalized Ayrton-Perry formula and the lateral-torsional buckling has been established and the correct meaning of the special imperfection factors has been deduced. Examining these factors the specialties of the formula have been revealed and suggestions for improvements have been made.

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